

# Conductance Correlations Near Integer Quantum Hall Transitions

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(February 1, 2008)

In a disordered mesoscopic system, the typical spacing between the peaks and the valleys of the conductance as a function of Fermi energy  $E_F$  is called the conductance energy correlation range  $E_c$ . Under the ergodic hypothesis, the latter is determined by the half-width of the ensemble averaged conductance correlation function:  $F = \langle \delta g(E_F) \delta g(E_F + \Delta E) \rangle$ . In ordinary diffusive metals,  $E_c \sim D/L^2$ , where  $D$  is the diffusion constant and  $L$  is the linear dimension of the phase-coherent sample. However, near a quantum phase transition driven by the location of the Fermi energy  $E_F$ , the above picture breaks down. As an example of the latter, we study, for the first time, the conductance correlations near the integer quantum Hall transitions of which  $E_F$  is a critical coupling constant. We point out that the behavior of  $F$  is determined by the interplay between the static and the dynamic properties of the critical phenomena.

PACS numbers: 73.50.Jt, 05.30.-d, 74.20.-z

The quantum interference effects in disordered phase-coherent systems belong to the mesoscopic physics [1,2]. A phase-coherent sample is one in which the phase-coherence length  $L_\phi$  is larger than the sample size  $L$ . Thus mesoscopic physics naturally appears in small systems of mesoscopic dimensions, usually in nanostructures. Mesoscopic physics is also important in systems large enough to exhibit macroscopic quantum phase transitions. The reason is that  $L_\phi$ , being the cutoff for the critical fluctuations, diverges as the temperature  $T$  approaches zero at such transitions. An important example of the latter is a two dimensional electron gas (2DEG) close to the transition between two quantized Hall plateaus. The mesoscopic fluctuations of the conductance in this case has been studied recently both experimentally and theoretically [3–9].

The essential physics in the mesoscopic regime is the lack of self-averaging in the transport properties. Sample specific, reproducible fluctuations in the conductance become observable at low temperatures. From a theoretical point of view, the basic statistical properties of conductance fluctuations are determined by the conductance correlation function [2],

$$F(E_F, \Delta E) = \langle \delta g(E_F + \Delta E) \delta g(E_F) \rangle, \quad (1)$$

where  $\delta g$  is the deviation from the impurity averaged conductance, *i.e.*  $\delta g(E_F) = g(E_F) - \langle g(E_F) \rangle$ . Note that the conductances in  $F$  are to be evaluated at *two different Fermi energies* separated by the amount  $\Delta E$ . In general,  $F$  can have another argument  $\Delta B$ , representing the magnetic field correlation of the conductance, which we shall not consider here.

For  $\Delta E = 0$ , Eq. (1) gives the variance of the conductance,  $\text{Var}(g) = F(0) = \langle \delta g^2 \rangle$ . In the rest of the paper, we measure the conductance in units of  $e^2/h$  and the variance in units of  $(e^2/h)^2$ . Under the ergodic hypothesis, the sample to sample fluctuations are analogous to the fluctuations of the conductance as a function of the Fermi energy. In this case, the typical spacing of

the peaks and valleys in the conductance as a function of energy in a specific sample, usually called the energy correlation range  $E_c$ , is determined by the half-width of the ensemble averaged conductance correlation function  $F(\Delta E)$  in Eq. (1), *i.e.*  $E_c = \Delta E_{1/2}$ .

In ordinary mesoscopic disordered metals in the diffusive regime, the disorder-averaged conductance,  $\langle g \rangle$ , can vary by orders of magnitude, but the variance of the conductance assumes a universal value of order one [2]. Moreover, the energy correlation range of the conductance is given by  $E_c \approx \hbar \pi^2 D / L^2$ , where  $D$  is the diffusion constant and  $L$  is the linear dimension of the system. It is important to emphasize that in the diffusive regime, the conductance correlation in Eq. (1) is only a function of the energy difference  $\Delta E$  and is independent of  $E_F$  [2].  $E_c$  in this case corresponds to the inverse diffusion time across the sample in the current direction which is often referred to as the Thouless energy. This is the characteristic Fermi energy difference beyond which the paths of two injected electrons cease to be phase coherent, giving rise to significant difference in the conductances.

However, as we shall show in this paper, in the critical regime of a quantum phase transition (QPT) that is *driven by the location of the Fermi energy* instead of correlation strengths, the above picture breaks down. The primary reason is that, in this case, the Fermi energy is a critical coupling constant that controls the proximity to a quantum critical point. As a result, the conductance correlation function in Eq. (1) is determined by the critical properties associated with the QPT. We shall focus on the QHE in which the transitions between the quantized Hall plateaus as a function of the magnetic field is driven by the location of the Fermi energy of the disordered 2D electron system.

It is well known that the quantum Hall transition (QHT) is a continuous zero temperature phase transition at a single extended state energy  $E^*$  between two adjacent Hall plateaus [10]. The critical singularity at the QHT is described by a single divergent length, the

localization length,  $\xi(E_F) \propto |E_F - E^*|^{-\nu}$ , as  $E_F$  approaches  $E^*$ . Here  $\nu$  is the localization length exponent. We shall focus on the critical regime, finite systems of linear dimension  $L$ , and the zero temperature limit. In this case, the critical fluctuations are cutoff by  $L$ . The width of the critical regime shrinks with increasing  $L$  according to  $L^{-1/\nu}$ . The transition thus acquires a finite width  $W \propto L^{-1/\nu}$ . The conductance in the transition regime is dominated by phase coherent transport and thus exhibits mesoscopic phenomena. What is different from ordinary diffusive metals is the proximity to the quantum critical point. If the typical spacings between the peaks and valleys in  $g(E_F)$ , *i.e.* the energy correlation range  $E_c \sim L^{-\alpha}$ , a large number of oscillations would appear within the critical region  $W$  so long as  $\alpha > 1/\nu \simeq 0.42$ , which will be shown to be the case below. In this regime, the ergodic hypothesis, which is expected to fail in the plateau phases, remains valid. Moreover, *the conductance correlations are determined by the critical properties associated with the QHT.*

We now proceed to write down the scaling form for the correlation function defined in Eq. (1) near the QHT,

$$F(E_F, \Delta E) = \mathcal{F} \left[ \frac{L}{\xi(E_F + \Delta E)}, \frac{L}{\xi(E_F)}, \frac{L}{L_\omega(\Delta E)} \right]. \quad (2)$$

Here  $L_\omega$  is the length scale introduced by a finite frequency. The origin of the latter is the following. In calculating the correlation function of the DC conductances at different energies, the energy difference  $\Delta E$  enters formally as a finite frequency. This was first pointed out by Lee, Stone, and Fukuyama [2] in their diagrammatic evaluation of  $F$  in diffusive metals. The easiest way to see that  $\xi$  must enter the scaling function is to consider  $\Delta E = 0$ , in which case, Eq. (2) gives the expected result [4]:  $F(E_F) = \langle \delta g^2(E_F) \rangle = \mathcal{F}[L/\xi(E_F)]$ . Moreover, both  $\xi(E_F + \Delta E)$  and  $\xi(E_F)$  must enter as scaling arguments in  $\mathcal{F}$ , because the range of  $\Delta E$  wherein  $\xi(E_F + \Delta E) \sim \xi(E_F)$  is given by  $\Delta E \ll \text{const} \times \xi(E_F)^{-(1+1/\nu)}$ , which is very small in the critical regime and vanishes much faster than the transition width  $W$ .

Eq. (2) shows that in general, in the critical regime of the QHT, both  $E_F$  and  $\Delta E$  enter the energy correlation function of the conductance. More important is the dual-role played by the Fermi energy difference. Writing  $\xi(E) \sim |E - E^*|^{-\nu}$  and  $L_\omega \sim |\Delta E|^{-1/z}$  with  $z$  the dynamical scaling exponent, and setting one of the Fermi energy  $E_F = E^* \equiv 0$ , Eq. (2) becomes,

$$F(\Delta E) = \mathcal{F} \left( |\Delta E| L^{1/\nu}, |\Delta E| L^z \right). \quad (3)$$

Eq. (3) clearly shows that  $\Delta E$  is a coupling constant conjugate to the static correlation (localization) length, and at the same time, a quantity analogous to a finite frequency conjugate to the length scale determined by the dynamical scaling exponent  $z$ . As a result, *both the static and the dynamic* critical properties enter the DC conductance correlation function. In general one expects

that Harris criteria  $\nu z > 1$  holds. The two scaling arguments in Eq. (3) compete and the correlation function must show a novel crossover from the regime dominated by static ( $\Delta E \sim L^{-1/\nu}$ ) fluctuations at large  $\Delta E$  to that dominated by dynamic ( $\Delta E \sim L^{-z}$ ) fluctuations at small  $\Delta E$ . Consequently, one expects the energy correlation range  $E_c$  to interpolate between  $E_c \sim L^{-1/\nu}$  at large  $\Delta E$  and  $E_c \sim L^{-z}$  at small  $\Delta E$ .

We next present a direct numerical calculation of the conductance correlation function  $F$  in Eq. (3) for an integer QHT in which the effects of electron-electron interactions are not considered [11]. In this theoretical *noninteracting analog* of the true integer QHT in real materials, it is known that  $\nu \simeq 2.3$  [10] and the dynamical exponent  $z = 2$ . That  $z = 2$  comes from the energy level spacing in a  $d = 2$  noninteracting electron system and is consistent with the (anomalous) diffusive dynamics known at the noninteracting integer QHT [12]. We will demonstrate that  $E_c$  indeed decays as  $L^{-1/\nu}$  at large  $\Delta E$  and as  $L^{-z}$  at small  $\Delta E$ . Remarkably, the crossover region between the latter two behaviors is rather broad in  $\Delta E$  over which we find  $E_c \sim L^{-1}$ .

For convenience, we choose to describe the transport in the integer quantum Hall regime using the Chalker-Coddington network model [16,17]. The latter is a square-lattice of potential saddle points (nodes) where quantum tunnelings between the edge states of the Hall droplets take place. With a choice of gauge [16], the transfer matrix at each node is given by,

$$T_{\text{node}} = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}, \quad (4)$$

with a single real parameter  $\theta$ . We have explicitly verified that introducing randomness in  $\theta$  does not change any of our results near the transition in agreement with the results of Ref. [17]. Away from the nodes, the edge electrons move along the links (equipotential contours) with a fixed chirality set by the direction of the magnetic field and accumulate random Bohm-Aharonov phases. Note that besides the distribution of these random link phases,  $\theta$  in Eq. (4) is the only parameter of the network. Changing  $\theta$  amounts to varying the Fermi energy  $E_F$  across the QHT. In the rest of the paper, we will present results in terms of  $E_F$  which is related to the parameter  $\theta$  by the choice of  $\theta = \theta_c \exp(E_F - E^*)$  and  $\sinh \theta_c = 1$  [17].

We have performed large scale numerical calculations of the two-terminal conductance. To this end, two semi-infinite ideal leads are attached to the left and right ends of the disordered network [4,5], and periodic or open boundary conditions are applied in the transverse direction. Let us consider disordered networks having  $L$  columns of nodes and  $W/2$  channels. Under such settings, the two-terminal conductance of a given sample with a fixed disorder realization is given by the Landauer formula [13],

$$g(E) = \frac{e^2}{h} \text{Tr}[\mathbf{t}^\dagger \mathbf{t}], \quad (5)$$

where  $\mathbf{t}$  is the  $(W/2) \times (W/2)$  transmission matrix. In the transfer matrix approach, it is convenient to express  $g$  in terms of the  $(W \times W)$  transfer matrix  $\mathbf{T}$  [14], since the latter is multiplicative across the  $L$  columns of scattering nodes in the network. Defining the ordered eigenvalues  $(\{\lambda_i\})$  of the symplectic matrix,  $H = \mathbf{T}^\dagger \mathbf{T}$ , by  $\lambda_i = \exp(2\gamma_i)$  for  $i = 1, \dots, W$ , Eq. (5) can be written as,

$$g(E) = \frac{e^2}{h} \sum_{i=1}^{W/2} \frac{1}{\cosh^2(\gamma_i(E))}. \quad (6)$$

Thus the calculation of the two-terminal conductance is transformed into that of the eigenvalues of the transfer matrix product  $H$ . It is known that constructing  $\mathbf{T}^\dagger \mathbf{T}$  by direct matrix multiplications is numerically unstable when the system size is large. We use here the stable numerical algorithm developed recently for large scale conductance calculations [4]. The details of this algorithm have been discussed in Refs. [9,15]. The basic idea is to maintain the stability of matrix multiplications using the method of matrix UDR-decomposition, and to extract the eigenvalues using the method of orthonormal projection. Specifically, one can show following a sequence of UDR-decompositions, the  $n$ -th power of  $H$ , can be written as  $H^n = U_n D_n R_n$ . In the limit of large  $n$ , typically less than 15, (i)  $U_n$  is a unitary matrix of which the columns converge to the eigenvectors of  $H$ ; (ii)  $D_n$  is a diagonal matrix and the eigenvalues of  $H$  is given by  $D_n D_{n-1}^{-1}$ ; and (iii)  $R_n$  converges to a limiting right triangular matrix with unity on the diagonals. We next present the numerical results obtained using this algorithm on networks with  $L = W$ , and  $L$  up to 128 in units of the lattice spacing.

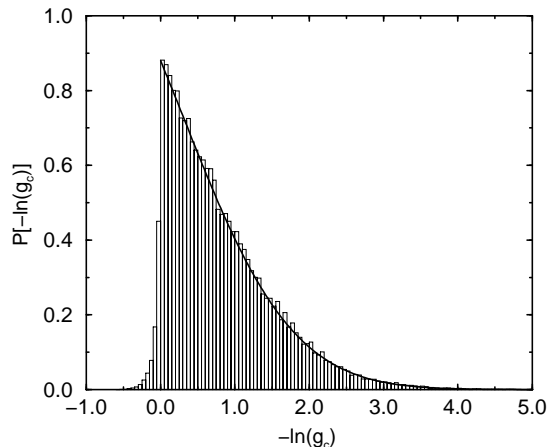


FIG. 1. Critical conductance distribution showing the skewed log-normal behavior (solid-line).

The critical conductance fluctuations at the QHT have been studied recently [4,5]. The critical conductance  $g_c$  was found to be broadly distributed between 0 and  $1e^2/h$  with log-normal characteristics of the central moments. Fig. 1 shows the distribution function of  $\log(g_c)$

calculated from Eq. (6) for 49,000 disorder realizations at  $L = 128$ . It shows a remarkable skewed log-normal behavior as a result of the sharp fall off of  $P[g_c]$  close to  $1e^2/h$ .

In Fig. 2, we plot the conductance as a function of the Fermi energy in a typical sample for four different sample sizes. It is important to note that these reproducible spectra exhibit remarkably smooth oscillations with well defined peaks and valleys in the transition regime. More oscillations appear with increasing system size. Since the transition width shrinks as  $L^{-1/\nu}$ , the typical spacing between the peaks and the valleys, *i.e.* the correlation range  $E_c$ , must decrease systematically with increasing  $L$  faster than as  $L^{-1/\nu}$ . These features are very different from the conductance fluctuation spectrum in diffusive metals of mesoscopic dimensions.

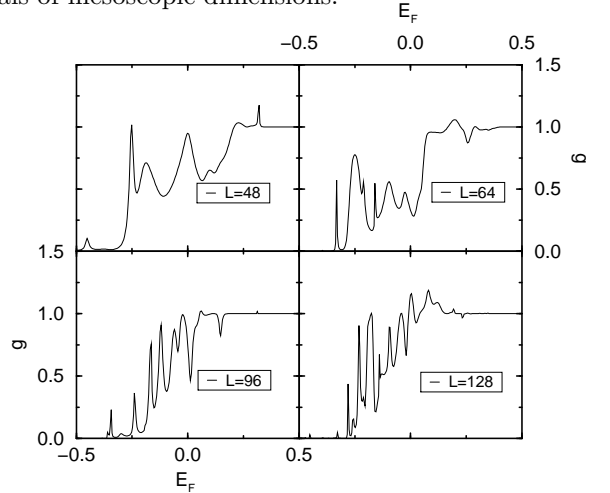


FIG. 2. Conductance fluctuation spectra in four typical samples of  $L = 48, 64, 96, 128$ .

From an ensemble of up to 50,000 samples for each  $L$ , having individual fluctuation spectrum exemplified in Fig. 2, we calculated the conductance correlation function,  $F(\Delta E)$ , defined in Eq. (3). In Fig. 3, we plot the normalized conductance correlation function by the variance of the critical conductance,  $F_n(\Delta E) \equiv F(\Delta E)/F(0)$ , as a function of  $\Delta E$  for different system sizes. The half widths  $\Delta E_{1/2}$  of each correlation function curve can be determined to extract the correlation range  $E_c$  versus  $L$ . Doing so, we obtained  $E_c \sim L^{-1}$ . To understand this rather surprising result, we now perform a scaling analysis of the correlation function in Eq. (3). Notice that the scaling function has two arguments originated from two different length scales,  $\xi \sim |\Delta E|^{-\nu}$  and  $L_\omega \sim |\Delta E|^{-1/z}$ . The scaling function should be dominated by the smaller one of the two. However, both of them are controlled by the same parameter,  $\Delta E$ , *i.e.* the difference in the Fermi energy around the critical point. For the noninteracting QHT,  $\nu \simeq 2.3$  and  $1/z = 0.5$ . Thus  $L_\omega \ll \xi$  for  $\Delta E \ll 1$  whereas  $\xi \ll L_\omega$  for  $\Delta E \gg 1$ . We therefore introduce a new length scale  $\lambda(\Delta E)$ , which interpolates between  $\xi(\Delta E)$  and  $L_\omega(\Delta E)$  as a function of  $\Delta E$ , and rewrite Eq. (3) as

$$F(\Delta E) = \mathcal{F}(\lambda(\Delta E)/L). \quad (7)$$

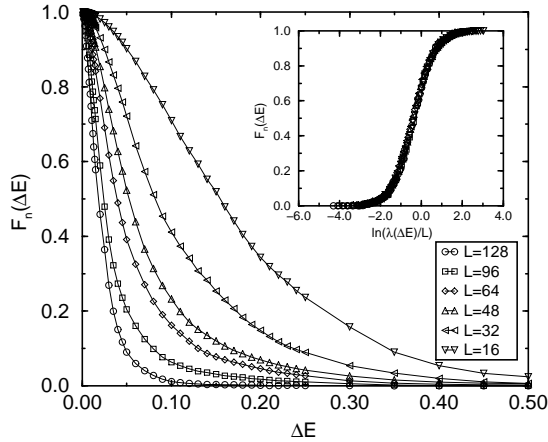


FIG. 3. The normalized conductance correlation function versus the Fermi energy difference. The inset shows the scaling plot of the function in Eq. (7).

The complete behavior of  $\lambda(\Delta E)$  is obtained by demanding that all finite size data of  $F_n$  at different  $L$  and  $\Delta E$  in Fig. 3 collapse onto a single scaling curve when  $F_n$  is plotted vs  $\lambda/L$ . Such a scaling plot is shown in the inset of Fig. 3 where the curve represents the scaling function  $\mathcal{F}$  in Eq. (7). To our knowledge this is the first demonstration of the scaling behavior of conductance correlations near a quantum phase transition.

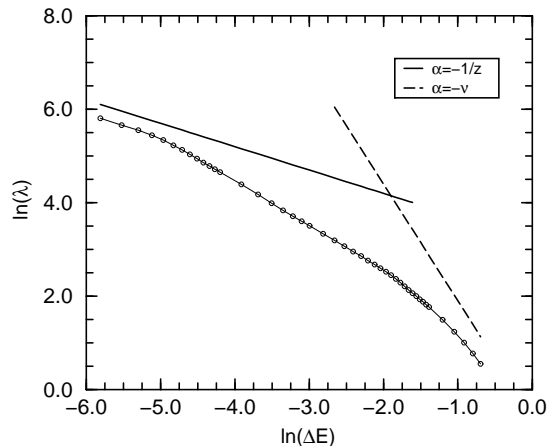


FIG. 4. The crossover length  $\lambda(\Delta E)$  versus  $\Delta E$  interpolating between static and dynamic fluctuation dominated asymptotics.

The obtained  $\lambda(\Delta E)$  shown in Fig. 4. Indeed,  $\lambda$  interpolates between the correct asymptotic behaviors dominated by the static and the dynamical critical fluctuations: (i) For small  $\Delta E$ ,  $\lambda \sim |\Delta E|^{-1/z}$  and consequently, the correlation energy  $E_c \sim L^{-z}$ . (ii) For large  $\Delta E$ ,  $\lambda \sim |\Delta E|^{-\nu}$ , implying  $E_c \sim L^{-1/\nu}$ . (iii) As Fig. 4 shows, the crossover regime between the asymptotic limits is

very broad. Remarkably, over almost the entire crossover region,  $\lambda$  exhibits a well defined power-law,

$$\lambda \sim |\Delta E|^{-1/\zeta}, \quad \zeta = 0.96 \pm 0.05. \quad (8)$$

As a result, the correlation energy  $E_c \sim L^{-1}$  over this broad region. This is consistent with the conclusion drawn from the half-width analysis of Fig. 3, and provides the microscopic mechanism by which such an unusual phenomenon takes place.

In summary, we have studied the energy correlation function of the conductance, a central quantity in mesoscopic physics, in the integer QHE. Although we have focused on the QHT in our numerical calculations, the physics discussed here is quite general and pertains to the critical regimes of quantum phase transitions that are driven by the location of the Fermi energy such as the metal-insulator transitions in disordered electronic systems.

The authors thank David Cobden, Dung-Hai Lee, Nick Read, Subir Sachdev, and Stuart Trugman for useful discussions, and Aspen center for physics for hospitality. This work is supported in part by an award from Research Corporation.

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